CONSTRUCTIONS OF COUPLING PROCESSES FOR LÉVY PROCESSES

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Abstract. We construct optimal Markov couplings of Lévy processes, whose Lévy (jump) measure has an absolutely continuous component. The construction is based on properties of subordinate Brownian motions and the coupling of Brownian motions by reflection.

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1. Introduction and Main Results

It is well known that a Lévy process $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ can be decomposed into three independent parts, i.e. the Gaussian part, the drift part and the jump part. The corresponding symbol or characteristic exponent (see [12, 15]) of X_t is given by

$$\psi(\xi) = \frac{1}{2}\langle Q\xi, \xi\rangle + i\langle b, \xi\rangle + \int_{z\neq 0}\Big(1 - e^{-i\langle \xi, z\rangle} + i\langle \xi, z\rangle \mathbb{1}_{\{|z|\leq 1\}}\Big)\nu(dz),$$

where $Q = (q_{j,k})_{j,k=1}^d$ is a positive semi-definite matrix, $b \in \mathbb{R}^d$ is the drift vector and ν is the Lévy or jump measure; the Lévy measure ν is a σ-finite measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{z\neq 0}(1 \wedge |z|^2)\nu(dz) < \infty$. If the matrix Q is strictly positive definite, regularity properties for the semigroup of a Lévy process can be easily derived from that of Brownian motion. However, when a Lévy process only has a pure jump part (i.e. $Q = 0$ and $\nu \neq 0$), the situation is completely different and, in general, more difficult to deal with. As a continuation of our recent work [17], we aim to construct optimal Markov coupling processes of Lévy process X_t, by assuming that the corresponding Lévy measure has absolutely continuous lower bounds.

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It has been proven in [19, Theorem 3.1] and [17, Theorem 1.1] that under some mild conditions compound Poisson processes admit successful couplings, and the corresponding transition probability function satisfies

$$\|P_t(x,\cdot)-P_t(y,\cdot)\|_{\mathrm{Var}}\le \frac{C(1+|x-y|)}{\sqrt{t}}\wedge 2\qquad \text{for } t>0 \text{ and } x,y\in\mathbb{R}^d, \tag{1.1}$$

where $\|\mu\|_{\mathrm{Var}}$ denotes the total variation norm of the signed measure μ; moreover, the factor $\sqrt{t^{-1}}$ in the inequality (1.1) is sharp for $t>0$ large enough. The following question is natural: *Is the rate $\sqrt{t^{-1}}$ also optimal for general Lévy processes that possess the coupling property*? Note that the Lévy measure ν is always finite outside a neighborhood of 0. Thus, the behavior of ν around the origin will be crucial for optimal estimates of $\|P_t(x,\cdot)-P_t(y,\cdot)\|_{\mathrm{Var}}$ as t tends to infinity.

Before stating our main results, we first present some necessary notations. A nonnegative function f on $(0,\infty)$ is called a *Bernstein function* if $f\in C^\infty(0,\infty)$, $f\ge 0$ and for all $k\ge 1$, $(-1)^k f^{(k)}(x)\le 0$. Any Bernstein function f has a Lévy-Khintchine representation

$$f(\lambda)=a+b\lambda+\int_0^\infty (1-e^{-\lambda s})\mu(ds),\qquad \lambda>0, \tag{1.2}$$

where a, $b>0$ and μ is a Radon measure on $(0,\infty)$ such that $\int_0^\infty (s\wedge 1)\mu(ds)<\infty$. In particular, the Lévy triplet (a,b,μ) determines the Bernstein function f uniquely and vice versa, e.g. see [16, Theorem 3.2].

Theorem 1.1. *Let X_t be a Lévy process on $\mathbb{R}^d$ and ν be its Lévy measure. Assume that*

$$\nu(dz)\ge |z|^{-d}f(|z|^{-2})dz, \tag{1.3}$$

where f is a Bernstein function. Then, there is a constant $C>0$ such that for x, $y\in\mathbb{R}^d$ and $t>0$,

$$\|P_t(x,\cdot)-P_t(y,\cdot)\|_{Var}\le \left(\frac{|x-y|}{\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{r}}e^{-ctf(r)}dr\right)\wedge \frac{C(1+|x-y|)}{\sqrt{t}}\wedge 2, \tag{1.4}$$

where $c=\pi^{d/2}\cos 1\big/(2d\Gamma(d/2+1))$.

Since $\int\big(1\wedge|z|^2)\nu(dz)<\infty$, we have $\int(1\wedge|z|^2)|z|^{-d}f(|z|^{-2})dz<\infty$. That is,

$$\int_0^1\frac{f(r)}{r}dr+\int_1^\infty\frac{f(r)}{r^2}dr<\infty,$$

which implies that the Bernstein function f in (1.3) should be without drift and killing terms (i.e. in the representation (1.2) we have $a=b=0$). Based on the coupling of random walks, we proved in [17, Corollaries 4.2 and 4.4] that any Lévy process, which is either strong Feller or whose Lévy measure has an absolutely continuous component, has the coupling property and (1.1) holds. Thus, (1.3) yields that (1.1) is valid in our setting. That is, the key and novel statement of Theorem 1.1 is the first term on the right hand side of the estimate (1.4). Note that for any x, $y\in\mathbb{R}^d$ and $t\ge 0$, $\|P_t(x,\cdot)-P_t(y,\cdot)\|_{\mathrm{Var}}\le 2$, and $\|P_t(x,\cdot)-P_t(y,\cdot)\|_{\mathrm{Var}}$ is decreasing with respect to t.

Hence the asymptotic of $\|P_t(x,\cdot)-P_t(y,\cdot)\|_{\mathrm{Var}}$ as $t\to\infty$ is more interesting. Obviously, $\int_0^\infty \frac{1}{\sqrt{r}}e^{-ctf(r)}dr<\infty$ for some constant $c>0$ and $t>0$ large enough if $\liminf_{r\to\infty}\frac{f(r)}{\log r}>0$. Indeed, we have

Proposition 1.2. *Assume that condition* (1.3) *holds. Then, for any $x,\ y\in\mathbb{R}^d$, as $t\to\infty$,*

$$\|P_t(x,\cdot)-P_t(y,\cdot)\|_{Var}=\begin{cases}\mathsf{O}\Big(\int_0^\infty \frac{1}{\sqrt{r}}e^{-ctf(r)}dr\Big), & f'(0+)=\infty;\\ \mathsf{O}(t^{-1/2}), & f'(0+)<\infty.\end{cases}\tag{1.5}$$

We will see from the next section that the assertion (1.5) is sharp in many situations. Here we only present a typical example to show the efficiency of Theorem 1.1.

Example 1.3. Assume that the Lévy measure satisfies

$$\nu(dz)\ge c|z|^{-d-\alpha}dz$$

for $c>0$ and $\alpha\in(0,2)$. Then by Theorem 1.1, for the associated Lévy process X_t, there exists a constant $C>0$ such that for any $x,\ y\in\mathbb{R}^d$ and $t>0$,

$$\|P_t(x,\cdot)-P_t(y,\cdot)\|_{\mathrm{Var}}\le\frac{C|x-y|}{t^{1/\alpha}}.$$

The main idea of the proof of Theorem 1.1 is to construct coupling processes of subordinate Brownian motions, by making full use of the coupling of Brownian motions by reflection. We will see in the next section that tail estimates for the coupling time of those coupling processes heavily depend on the decay of the associated Bernstein function $f(\lambda)$ as $\lambda\to0$. A number of examples are also presented to illustrate the optimality of such coupling processes for subordinate Brownian motions. The proofs and some comments of Theorem 1.1 and Proposition 1.2 are given in Section 3.

2. Couplings of Subordinate Brownian Motions

In this section, we will study the coupling property of a class of special but important Lévy processes—subordinate Brownian motions. Examples of subordinate Brownian motions include rotationally invariant stable Lévy processes, relativistic stable Lévy processes and so on.

Suppose that $(B_t)_{t\ge0}$ is a Brownian motion on $\mathbb{R}^d$ with

$$\mathbb{E}\Big[e^{i\xi(B_t-B_0)}\Big]=e^{-t|\xi|^2},\qquad \xi\in\mathbb{R}^d,t>0,$$

and $(S_t)_{t\ge0}$ is a subordinator (that is, $(S_t)_{t\ge0}$ is a nonnegative Lévy process such that S_t is increasing and right-continuous in t with $S_0=0$) independent of $(B_t)_{t\ge0}$. For any $t\ge0$, let μ_t^S be the transition probabilities of the subordinator S, i.e. $\mu_t^S(B)=\mathbb{P}(S_t\in B)$ for any $B\in\mathscr{B}([0,\infty))$. It is well known that the associated Laplace transformation of μ_t^S is given by

$$\int_0^\infty e^{-\lambda s}\mu_t^S(ds)=e^{-tf(\lambda)},\qquad \lambda>0,$$

where $f(\lambda)$ is a Bernstein function. We refer to [16] for more details about Bernstein functions and subordinators. Any subordinate Brownian motion $(X_t)_{t\geq 0}$ defined by $X_t = B_{S_t}$ is a symmetric Lévy process with

$$\mathbb{E}\Big[e^{i\xi(X_t - X_0)}\Big] = e^{-tf(|\xi|^2)}, \qquad \xi \in \mathbb{R}^d, t > 0.$$

That is, the symbol or characteristic exponent of subordinate Brownian motion X_t is $f(|\xi|^2)$, see [12].

Recall that the pair (X_t, X_t') is said to be a *coupling of the Markov process* X_t, if $(X_t')_{t\geq 0}$ is a Markov process such that it has same transition distribution as $(X_t)_{t\geq 0}$ but possibly different initial distributions. In this case, X_t and X_t' are called the *marginal processes* of the coupling process, and the coupling time is defined by $T := \inf\{t \geq 0 : X_t = X_t'\}$. The coupling (X_t, X_t') is called *successful* if T is finite. A Markov process is said to have successful couplings (or to have the coupling property) if for any two initial distributions μ_1 and μ_2, there exists a successful coupling with marginal processes starting from μ_1 and μ_2 respectively. In particular, according to [14] and the proof of [17, Theorem 4.1], the coupling property is equivalent to the statement that:

$$\lim_{t\to\infty} \|P_t(x, \cdot) - P_t(y, \cdot)\|_{\mathrm{Var}} = 0 \qquad \text{for any } x, y \in \mathbb{R}^d,$$

where $P_t(x, \cdot)$ is the transition function of marginal process. A Markov coupling process is called *optimal* if it can give us sharp estimates of $\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\mathrm{Var}}$ as t tends to infinity. The notion of optimal Markov coupling processes used here is different from the one used by [5, Definition 2.24].

To construct an optimal Markov coupling process of subordinate Brownian motion $(X_t)_{t\geq 0}$, we begin with reviewing known facts about the coupling of Brownian motions by reflection, see [13, 4, 11]. Fix x, $y \in \mathbb{R}^d$ with $x \neq y$. Let B_t^x be a Brownian motion on $\mathbb{R}^d$ $(d \geq 1)$ starting from $x \in \mathbb{R}^d$, and $H_{x,y}$ be the hyperplane such that the vector $x - y$ is normal with respect to $H_{x,y}$ and $(x + y)/2 \in H_{x,y}$, i.e.

$$H_{x,y} = \big\{u \in \mathbb{R}^d : \langle u - (x + y)/2, x - y\rangle = 0\big\}.$$

Denote by $R_{x,y} : \mathbb{R}^d \to \mathbb{R}^d$ the reflection with respect to the hyperplane $H_{x,y}$. Then, for every $z \in \mathbb{R}^d$,

$$R_{x,y}z = z - 2\langle z - (x + y)/2, x - y\rangle(x - y)/|x - y|^2.$$

Define

$$\tau_{x,y} = \inf\big\{t > 0 : B_t^x \in H_{x,y}\big\}$$

and

$$\hat{B}_t^y := \begin{cases} R_{x,y}B_t^x, & t \leq \tau_{x,y}; \\ B_t^x \ , & t > \tau_{x,y}. \end{cases}$$

That is, $\hat{B}_t^y$ is the mirror reflection of B_t^x with respect to $H_{x,y}$ before $\tau_{x,y}$ and coincides with B_t^x afterwards. It is clear that $\hat{B}_t^y$ is a Brownian motion starting from y. Set $\tilde{B}_t^{x,y} := (B_t^x, \hat{B}_t^y)$. Then, $\tilde{B}_t^{x,y}$ is a coupling of two Brownian motions starting from x, $y \in \mathbb{R}^d$ respectively. The coupling time

$$T_{x,y}^B := \inf\{t > 0 : B_t^x = \hat{B}_t^y\} \tag{2.6}$$

is just the stopping time $\tau_{x,y}$. By [4, Section 5, Page 170], we have

$$\mathbb{P}(T^B_{x,y} > t) = \sqrt{\frac{2}{\pi}} \int_0^{|x-y|/(2\sqrt{2t})} \exp\big(-u^2/2\big)du \le \frac{|x-y|}{2\sqrt{\pi t}}. \tag{2.7}$$

Note that B_t here is just the usual standard Brownian motion but running at twice the speed, so the factor $\sqrt{2}$ appears in the upper bound of integral in (2.7).

Next, let $(S_t)_{t\ge 0}$ be a subordinator with $S_0 = 0$, which is independent of $\tilde{B}^{x,y}_t$. Set

$$\tilde{X}^{x,y}_t = \tilde{B}^{x,y}_{S_t} = (B^x_{S_t}, \hat{B}^y_{S_t}).$$

Since $S_0 = 0$, according to the definition of subordinate Brownian motion, we get that $\tilde{X}^{x,y}_t$ is a coupling process of X_t starting from x and y. For simplicity, let $\tilde{X}^{x,y}_t := (X^x_t, \hat{X}^y_t)$, and call $\tilde{X}^{x,y}_t$ the *reflection-subordinate coupling* of X_t. Define the coupling time of $\tilde{X}^{x,y}_t$ as follows

$$T^X_{x,y} := \inf\{t \ge 0 : X^x_t = \hat{X}^y_t\}. \tag{2.8}$$

For any x, $y \in \mathbb{R}^d$, we will claim that $T^X_{x,y} < \infty$ almost surely. More precisely, we have

Theorem 2.1. *Let X_t be a subordinate Brownian motion on $\mathbb{R}^d$ corresponding to the Bernstein function f, and $P^f_t(x,\cdot)$ be its transition function. Then, X_t has the coupling property; moreover, for any $t > 0$ and x, $y \in \mathbb{R}^d$,*

$$\|P^f_t(x,\cdot) - P^f_t(y,\cdot)\|_{Var} \le 2\mathbb{P}(T^X_{x,y} > t) \le \frac{|x-y|}{\sqrt{2}\pi} \int_0^\infty \frac{1}{\sqrt{r}} e^{-tf(r)}dr. \tag{2.9}$$

Additionally, assume that $\liminf_{r\to\infty} f(r)/\log r > 0$, $\liminf_{r\to 0} f(r)|\log r| < \infty$ and that f^{-1} satisfies the following volume doubling property:

$$\limsup_{s\to 0} f^{-1}(2s)/f^{-1}(s) < \infty. \tag{2.10}$$

Then, there exists a constant $C > 0$ such that for $t > 0$ sufficiently large

$$\|P^f_t(x,\cdot) - P^f_t(y,\cdot)\|_{Var} \le C|x-y|\sqrt{f^{-1}\Big(\frac{1}{t}\Big)}. \tag{2.11}$$

Remark 2.2. (i) We mention that *if there exists $c > 0$ such that for $s > 0$ small enough, $2f(s) \le f(cs)$, then* (2.10) *holds.* Indeed, suppose that there exists $c_0 > 0$ such that $2f(s) \le f(cs)$ holds for all $s \in (0, c_0]$. By the monotonicity of f, $f^{-1}(2f(s)) \le cs$. That is, $\limsup_{s\to 0} f^{-1}(2f(s))/s \le c$. Since $f^{-1}(s) \to 0$ as $s \to 0$, we have $\limsup_{s\to 0} f^{-1}(2s)/f^{-1}(s) \le c$, and so (2.10) follows.

(ii) It is clear that if $\liminf_{r\to\infty} f(r)/\log r > 0$, then $\int_0^\infty \frac{1}{\sqrt{r}} e^{-tf(r)}dr < \infty$ for $t > 0$ large enough. We will claim that the converse is also true. Indeed, assume that $\int_0^\infty \frac{1}{\sqrt{r}} e^{-t_0 f(r)}dr < \infty$. Since $r \mapsto \frac{1}{\sqrt{r}} e^{-t_0 f(r)}$ is strictly decreasing on $[0,\infty)$, by a standard Abelian argument, there exist positive constants r_0 and c such that for any $r \ge r_0$,

$$\frac{1}{\sqrt{r}} e^{-t_0 f(r)} \le \frac{c}{r}.$$

That is, $f(r)/\log r \ge c/(2t_0)$. So, $\liminf_{r\to\infty} f(r)/\log r \ge c/(2t_0)$.

Before we prove Theorem 2.1 we give some examples. Here, we always suppose that S is a subordinator corresponding to the Bernstein function f, and X is the associated subordinate Brownian motion. Denote by $P_t^f(x,\cdot)$ the transition function of X. For two non-negative functions g and h, the notation $g \asymp h$ means that there are two positive constants c_1 and c_2 such that $c_1 g \le h \le c_2 g$. An extensive list of examples of Bernstein functions can be found in [16, Chapter 15].

Example 2.3. Consider $\alpha \in (0,2)$ and define

$$f(\lambda) = \lambda^{\alpha/2}.$$

The corresponding subordinate Brownian motion X_t is the rotationally invariant stable Lévy process with index α. In this case, for $t > 0$ sufficiently large, the estimate (2.11) becomes

$$\|P_t^f(x,\cdot) - P_t^f(y,\cdot)\|_{\mathrm{Var}} \le \frac{C|x-y|}{t^{1/\alpha}}. \tag{2.12}$$

On the other hand, let Z_t be a rotationally invariant α-stable process on $\mathbb{R}^d$ starting from 0. For any x, $y \in \mathbb{R}^d$ with $x < y$, i.e. $x_i < y_i$ for $1 \le i \le d$,

$$\begin{aligned}\|P_t^f(x,\cdot) - P_t^f(y,\cdot)\|_{\mathrm{Var}} \ge & \Big|\mathbb{P}\big(Z_t + x \in \Pi_{i=1}^d (x_i,\infty)\big) - \mathbb{P}\big(Z_t + y \in \Pi_{i=1}^d (x_i,\infty)\big)\Big| \\ = & \Big|\mathbb{P}\big(Z_t \in (0,\infty)^d\big) - \mathbb{P}\big(Z_t \in \Pi_{i=1}^d (x_i - y_i,\infty)\big)\Big| \\ = & \mathbb{P}\big(Z_t \in \Pi_{i=1}^d (x_i - y_i,\infty) \setminus (0,\infty)^d\big) \\ \ge & \sum_{j=1}^d \mathbb{P}\big(Z_t \in (x_j - y_j, 0] \times (0,\infty)^{d-1}\big).\end{aligned}$$

Denote by p_t the density function of Z_t. It is well known, see e.g. [6, 1], that

$$p_t(z) \asymp t^{-d/\alpha} \wedge \frac{t}{|z|^{d+\alpha}}.$$

Thus, for any $t \ge (y_1 - x_1)^\alpha$,

$$\begin{aligned}&\int_{(x_1-y_1,0]\times(0,\infty)^{d-1}} p_t(z)\,dz \\ \ge & c_0 \int_{(x_1-y_1,0]\times(0,\infty)^{d-1}} \left(t^{-d/\alpha} \wedge \frac{t}{|z|^{d+\alpha}}\right) dz \\ \ge & c_0 \int_{\sum_{i=2}^d z_i^2 \ge t^{2/\alpha}, z_i>0,\, i=2,\cdots,d} \int_{x_1-y_1}^0 \left[\frac{t}{\big((y_1-x_1)^2 + \sum_{i=2}^d z_i^2\big)^{(d+\alpha)/2}}\right] dz_1 dz_2 \cdots dz_d \\ \ge & c_1 (y_1 - x_1) t \int_{\sum_{i=2}^d z_i^2 \ge t^{2/\alpha}, z_i>0,\, i=2,\cdots,d} \frac{1}{\big(\sum_{i=2}^d z_i^2\big)^{(d+\alpha)/2}} dz_2 \cdots dz_d \\ = & c_2 (y_1 - x_1) t \int_{t^{1/\alpha}}^\infty \frac{1}{r^{2+\alpha}} dr \\ = & \frac{c_2 (y_1 - x_1)}{t^{1/\alpha}}.\end{aligned}$$

Therefore, for $t \geq \max_{i=1}^d (y_i - x_i)^\alpha$,

$$\begin{aligned}\|P_t^f(x,\cdot) - P_t^f(y,\cdot)\|_{\mathrm{Var}} \geq & \sum_{j=1}^d \int_{(x_j - y_j, 0]\times(0,\infty)^{d-1}} p_t(z)dz \\ \geq & \frac{c_2 \sum_{j=1}^d (y_j - x_j)}{t^{1/\alpha}} \\ \geq & \frac{c_2|y-x|}{t^{1/\alpha}}.\end{aligned}$$

This implies that (2.12) is sharp.

Example 2.4. Consider $0 < \alpha < \beta < 2$ and define

$$f(\lambda) = \lambda^{\alpha/2} + \lambda^{\beta/2}.$$

The corresponding subordinate Brownian motion X_t is a mixture of rotationally invariant stable Lévy processes with index α and β. For this example, for $t > 0$ large enough,

$$\sqrt{f^{-1}\left(\frac{1}{t}\right)} \asymp \frac{1}{t^{1/\alpha}},$$

and so

$$\|P_t^f(x,\cdot) - P_t^f(y,\cdot)\|_{\mathrm{Var}} \leq \frac{C|x-y|}{t^{1/\alpha}}.$$

That is, the degree of decay of $\|P_t^f(x,\cdot) - P_t^f(y,\cdot)\|_{\mathrm{Var}}$ (as t tends to infinity) is determined by the smaller index α. One also can see this assertion in the following way: Let $P_t^{(\alpha)}$ and $P_t^{(\beta)}$ be the semigroups corresponding to subordinate Brownian motions with Bernstein functions $f^{(\alpha)}(r) = r^{\alpha/2}$ and $f^{(\beta)}(r) = r^{\beta/2}$, respectively. According to the proof of Proposition 2.9 below, we have

$$\|P_t^f(x,\cdot) - P_t^f(y,\cdot)\|_{\mathrm{Var}} \leq \|P_t^{(\alpha)}(x,\cdot) - P_t^{(\alpha)}(y,\cdot)\|_{\mathrm{Var}} \wedge \|P_t^{(\beta)}(x,\cdot) - P_t^{(\beta)}(y,\cdot)\|_{\mathrm{Var}}.$$

Then, the desired assertion follows from Example 2.3 above.

Recently it has been proven in [7, Theorem 1.2] (see also [9]) that the density function of a mixture of rotationally invariant stable Lévy processes with index α and β satisfies

$$p(t,x,y) \asymp \left(t^{-d/\alpha} \wedge t^{-d/\beta}\right) \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{t}{|x-y|^{d+\beta}}\right)$$

on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. Since $\alpha < \beta$, for $t > 0$ large enough, $t^{-d/\alpha} < t^{-d/\beta}$, and so there exists $c > 0$ such that for $t > 0$ large enough,

$$p(t,x,y) \geq c\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

The right hand side of the inequality above is just the sharp estimate (up to a constant) of the density function of rotationally invariant α-stable Lévy process. This implies that for this example our upper bound $t^{-1/\alpha}$ is optimal for $t > 0$ large enough, cf. also Example 2.3.

The following two Bernstein functions are taken from [18, Chapter 5.2.2; Examples 2.15 and 2.16].

Example 2.5. Consider $0 < \alpha < 2$, $\beta \in (0, 2-\alpha)$ and define

$$f(\lambda) = \lambda^{\alpha/2}(\log(1+\lambda))^{\beta/2}.$$

Noting that $f(\lambda) \asymp \lambda^{(\alpha+\beta)/2}$ as $\lambda \to 0$, for the corresponding subordinate Brownian motion,

$$\|P_t^f(x,\cdot) - P_t^f(y,\cdot)\|_{\mathrm{Var}} \le \frac{C|x-y|}{t^{1/(\alpha+\beta)}} \qquad \text{for } t > 0 \text{ large enough.}$$

Example 2.6. Consider $0 < \alpha < 2$, $\beta \in (0, \alpha)$ and define

$$f(\lambda) = \lambda^{\alpha/2}(\log(1+\lambda))^{-\beta/2}.$$

Since $f(\lambda) \asymp \lambda^{(\alpha-\beta)/2}$ as $\lambda \to 0$, we know that in this situation, for $t > 0$ large enough,

$$\|P_t^f(x,\cdot) - P_t^f(y,\cdot)\|_{\mathrm{Var}} \le \frac{C|x-y|}{t^{1/(\alpha-\beta)}}.$$

As we can see from (2.11) and, in particular, by the four examples from above, estimates about $\|P_t^f(x,\cdot) - P_t^f(y,\cdot)\|_{\mathrm{Var}}$ depend on the decay of $f(\lambda)$ as λ tends to zero. Roughly speaking, the smaller $f(\lambda)$ near zero, the larger is the upper bound in (2.11). The following three examples further illustrate this point. They also show that under the reflection-subordinate coupling the coupling happens not necessarily faster than under (compound) Poissonian coupling, cf. (1.1) and [17, Theorem 1.1]).

Example 2.7. Consider $0 < \alpha < 2$, $m > 0$ and define

$$f(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m.$$

We state that as $\lambda \to 0$, $f(\lambda) \asymp \lambda$. The corresponding subordinate process is the relativistic stable Lévy process. For $t > 0$ large enough,

$$\|P_t^f(x,\cdot) - P_t^f(y,\cdot)\|_{\mathrm{Var}} \le \frac{C|x-y|}{\sqrt{(m+t^{-1})^{2/\alpha} - m^{2/\alpha}}} \asymp \frac{C|x-y|}{\sqrt{t}}.$$

The estimate above is sharp for $t > 0$ large enough. Indeed, for $m = \alpha = d = 1$, it can be shown that (e.g. see [10, 3] or [7, Example 2.4]) for every $t > 0$ and $(x, y) \in \mathbb{R} \times \mathbb{R}$, the density function of relativistic stable Lévy process satisfies

$$p(t,x,y) \ge \frac{c_1 t}{(|x-y|+t)^2}\left(1 \vee (|x-y|+t)^{1/2}\right) e^{-c_2 \frac{|x-y|^2}{\sqrt{|x-y|^2+t^2}}}.$$

In particular, for $t > 0$ large enough, we have

$$p(t,x,y) \ge \frac{c_1 t}{(|x-y|+t)^{3/2}} e^{-c_2 \frac{|x-y|^2}{\sqrt{|x-y|^2+t^2}}}.$$

Let Z_t be a relativistic stable Lévy process with $m = \alpha = 1$ on $\mathbb{R}$ starting from 0. Then, for any x, $y \in \mathbb{R}$ with $x < y$ and $t > 0$ large enough,

$$\begin{aligned}\|P_t^f(x,\cdot) - P_t^f(y,\cdot)\|_{\mathrm{Var}} \geq & \big|\mathbb{P}\big(Z_t + x \in (x,\infty)\big) - \mathbb{P}\big(Z_t + y \in (x,\infty)\big)\big| \\ = & \big|\mathbb{P}\big(Z_t \in (0,\infty)\big) - \mathbb{P}\big(Z_t \in (x-y,\infty)\big)\big| \\ = & \mathbb{P}\big(Z_t \in (x-y,0]\big) = \int_{x-y}^0 p(t,0,z)dz \\ \geq & c_1 t \int_{x-y}^0 \frac{1}{(|u|+t)^{3/2}} e^{-c_2 \frac{u^2}{\sqrt{u^2+t^2}}} du \\ \geq & \frac{C_1(y-x)}{t^{1/2}}.\end{aligned}$$

Example 2.8. First, we consider $0 < \alpha \leq 1$ and define

$$f(\lambda) = \log^{1/\alpha}(1+\lambda^\alpha),$$

which satisfies that $f(\lambda) \asymp \lambda$ as $\lambda \to 0$. When $\alpha = 1$, S_t is called Gamma subordinator. In this setting, for $t > 0$ large enough,

$$\|P_t^f(x,\cdot) - P_t^f(y,\cdot)\|_{\mathrm{Var}} \leq C|x-y|\Big(\exp(t^{-\alpha}) - 1\Big)^{1/(2\alpha)} \asymp \frac{C|x-y|}{\sqrt{t}}.$$

On the other hand, we study the coupling property of rotationally invariant geometric stable Lévy processes, which are subordinate Brownian motions associated with the Bernstein function $g(\lambda) = \log(1+\lambda^\alpha)$ and $0 < \alpha \leq 2$. One can see that for these processes, when $t > 0$ large enough,

$$\|P_t^g(x,\cdot) - P_t^g(y,\cdot)\|_{\mathrm{Var}} \leq \frac{C|x-y|}{t^{1/\alpha}}.$$

This assertion is the same as that for rotationally invariant stable Lévy processes, but completely different from Brownian motions subordinated with f. We furthermore point out that for rotationally invariant geometric stable Lévy processes, $g(\lambda) \asymp \lambda^\alpha$ as $\lambda \to 0$.

Now, we turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. We follow the proof of [17, Proposition 3.3] to verify that the relation between coupling times defined by (2.6) and (2.8) is

$$T_{x,y}^X = \inf\big\{t \geq 0 : S_t \geq T_{x,y}^B\big\}. \tag{2.13}$$

In the following argument, assume that ω is fixed, and define $K_{x,y} = \inf\big\{t \geq 0 : S_t \geq T_{x,y}^B\big\}$. Let $t > 0$ be such that $S_t \geq T_{x,y}^B$, i.e. $t \geq K_{x,y}$. Since $B_t^x = \hat{B}_t^y$ for $t \geq T_{x,y}^B$, we have $B_{S_t}^x = \hat{B}_{S_t}^y$, and, by construction, $X_t = \hat{X}_t$. That is, $T_{x,y}^X \leq t$. Since $t \geq K_{x,y}$ was arbitrary, we have $T_{x,y}^X \leq K_{x,y}$. On the other hand, assume that $K_{x,y} > 0$. Then, by the definition of $K_{x,y}$, for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that $t_\varepsilon > K_{x,y} - \varepsilon$ and $S_{t_\varepsilon} < T_{x,y}^B$. Hence, $B_{S_{t_\varepsilon}}^x \neq \hat{B}_{S_{t_\varepsilon}}^y$, i.e. $X_{t_\varepsilon}^x \neq \hat{X}_{t_\varepsilon}^y$. Therefore, $T_{x,y}^X \geq t_\varepsilon > K_{x,y} - \varepsilon$. Letting $\varepsilon \to 0$, we get $T_{x,y}^X \geq K_{x,y}$, and thus (2.13) holds.

Now, according to (2.7), for almost every ω we have $T^B_{x,y}(\omega) < \infty$. Since the subordinator S_t tends to infinity as $t \to \infty$, there exists $\tau_0(\omega) < \infty$ such that $S_t(\omega) \ge T^B_{x,y}(\omega)$ for all $t \ge \tau_0(\omega)$. Therefore, (2.13) implies that $T^X_{x,y} \le \tau_0 < \infty$.

For any x, $y \in \mathbb{R}^d$ and $t > 0$, by the classic coupling inequality, (2.13) and (2.7),

$$\begin{aligned}\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{Var}} &\le 2\mathbb{P}(T^X_{x,y} > t)\\ &= 2\mathbb{P}(T^B_{x,y} > S_t)\\ &= 2\int_0^\infty \mathbb{P}(T^B_{x,y} > s)\mathbb{P}(S_t \in ds)\\ &\le \frac{|x-y|}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{s}}\,\mu^S_t(ds).\end{aligned}$$

According to the fact that

$$\frac{1}{\sqrt{s}} = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{r}}e^{-rs}dr,$$

we obtain

$$\int_0^\infty \frac{1}{\sqrt{s}}\,\mu^S_t(ds) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{r}}\int_0^\infty e^{-rs}\mu^S_t(ds)dr = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{r}}e^{-tf(r)}dr,$$

which in turn gives us (2.9).

Since the Bernstein function f is strictly increasing, we can make a change of variables to get

$$\int_0^\infty \frac{1}{\sqrt{r}}e^{-tf(r)}dr = \int_0^\infty \frac{e^{-ts}}{\sqrt{f^{-1}(s)}f'(f^{-1}(s))}ds = 2\int_0^\infty e^{-ts}d\sqrt{f^{-1}(s)}.$$

Suppose that $\liminf\limits_{r\to\infty} f(r)/\log r > 0$ and (2.10) hold. Then, we can choose positive constants c_i ($i = 1, 2, 3$) such that $f^{-1}(2x) \le c_1 f^{-1}(x)$ if $x \in (0, 2c_3]$; $f^{-1}(x) \le e^{c_2 x}$ if $x \in [c_3, \infty)$. Thus, for $t > 0$ large enough,

$$\begin{aligned}\int_0^\infty e^{-ts}d\sqrt{f^{-1}(s)} &= e^{-ts}\sqrt{f^{-1}(s)}\Big|_0^\infty + \int_0^\infty f^{-1}\Big(\frac{s}{t}\Big)e^{-s}ds\\ &= \int_0^\infty f^{-1}\Big(\frac{s}{t}\Big)e^{-s}ds.\end{aligned}$$

For any $s \in (1, c_3 t]$, choose $k = [\log_2 s] + 1$. Since f^{-1} is increasing, we find

$$f^{-1}\Big(\frac{s}{t}\Big) \le f^{-1}\Big(\frac{2^k}{t}\Big) \le c_1^k f^{-1}\Big(\frac{1}{t}\Big) \le 2^{k\rho} f^{-1}\Big(\frac{1}{t}\Big) \le (2s)^\rho f^{-1}\Big(\frac{1}{t}\Big),$$

where $\rho = \log_2 c_1$. Therefore, for $t > 0$ large enough,

$$\begin{aligned}\int_0^\infty f^{-1}\Big(\frac{s}{t}\Big)e^{-s}ds &= \Big(\int_0^1 + \int_1^{c_3 t} + \int_{c_3 t}^\infty\Big) f^{-1}\Big(\frac{s}{t}\Big)e^{-s}ds\\ &\le f^{-1}\Big(\frac{1}{t}\Big) + 2^\rho f^{-1}\Big(\frac{s}{t}\Big)\int_1^{c_3 t} s^\rho e^{-s}ds + \int_{c_3 t}^\infty e^{-(s-c_2 s/t)}ds\\ &\le \Big[1 + 2^\rho\int_1^\infty s^\rho e^{-s}ds\Big] f^{-1}\Big(\frac{1}{t}\Big) + \int_{c_3 t}^\infty e^{-s/2}ds\\ &\le C_1 f^{-1}\Big(\frac{1}{t}\Big) + 2e^{-c_3 t/2}.\end{aligned}$$

Since $\liminf_{r\to 0} f(r)|\log r| < \infty$, there exist positive constants c_4 and r_0 such that for $r \le r_0$, $f(r) \le c_4 \big/ \log r^{-1}$. Thus, for $t > 0$ large enough, $f^{-1}(1/t) \ge \exp(-c_4 t)$. According to the volume doubling property (2.10) again, we get that for $t > 0$ large enough,

$$f^{-1}(1/t) \ge c_5 e^{-c_3 t/2}.$$

This along with all the above conclusions above yields the required assertion. $\square$

Theorem 2.1 is easily generalized to study the coupling property of Lévy processes, which can be decomposed into two independent parts, one of which is a subordinate Brownian motion.

Proposition 2.9. *Suppose that the Lévy process X_t can be split into*

$$X_t = Y_t + B_t^f,$$

where B_t^f is a Brownian motion subordinated by the subordinator S and Y_t is a Lévy process. Let $P_t(x,\cdot)$ be the transition probability function of X_t. Then, there exists a constant $C > 0$ such that for $t > 0$ and $x, y \in \mathbb{R}^d$,

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{Var} \le \left(\frac{|x-y|}{\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{r}}e^{-tf(r)}dr\right) \wedge \frac{C(1+|x-y|)}{\sqrt{t}} \wedge 2,$$

where $f(\lambda)$ is the Bernstein function corresponding to S.

Proof. Let P_t^f and P_t^Y be the semigroups of B_t^f and Y_t respectively. Then,

$$\begin{aligned}\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{Var}} &= \sup_{\|f\|_\infty \le 1} \big|P_t f(x) - P_t f(y)\big|\\ &= \sup_{\|f\|_\infty \le 1} \big|P_t^f P_t^Y f(x) - P_t^f P_t^Y f(y)\big|\\ &\le \sup_{\|h\|_\infty \le 1} \big|P_t^f h(x) - P_t^f h(y)\big|\\ &= \|P_t^f(x,\cdot) - P_t^f(y,\cdot)\|_{\mathrm{Var}}.\end{aligned}$$

Note that the Lévy measure of any subordinate Brownian motion is absolutely continuous with respect to the Lebesgue measure. According to [17, Theorem 4.3 and Corollary

4.4], there exists a constant $C>0$ such that for any $t>0$, x, $y\in\mathbb{R}^d$,

$$\|P_t^f(x,\cdot)-P_t^f(y,\cdot)\|_{\mathrm{Var}}\le \frac{C(1+|x-y|)}{\sqrt{t}}\wedge 2.$$

Combining this with Theorem 2.1 yields

$$\|P_t^f(x,\cdot)-P_t^f(y,\cdot)\|_{\mathrm{Var}}\le \left(\frac{|x-y|}{\sqrt{2}\pi}\int_0^\infty \frac{1}{\sqrt{r}}e^{-tf(r)}dr\right)\wedge \frac{C(1+|x-y|)}{\sqrt{t}}\wedge 2.$$

We have proved the desired assertion. □

Remark 2.10. Proposition 2.9 can be stated in the following way: *Let $\Phi(\xi)$ be the symbol of Lévy process X_t. If $\Phi(\xi)=f(|\xi|^2)+\Psi(\xi)$, where f is a Bernstein function and $\Psi(\xi)$ is again a symbol of a Lévy process, then the conclusion of Proposition 2.9 holds.*

3. Proof and Extension of Theorem 1.1

We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. We first suppose that the Lévy measure of X_t satisfies

$$\nu(dz)\ge c|z|^{-d}f(|z|^{-2})dz, \tag{3.14}$$

where

$$c=\left(\int_{\{|z|\le 1\}}(1-\cos z_1)|z|^{-d}dz\right)^{-1}.$$

Then, by the argument of [20, Theorem 1.1, Part (a)],

$$X_t=X_t'+B_t^f,$$

where B_t^f is a subordinated Brownian motion corresponding to the Bernstein function f and X_t' is a Lévy process with symbol

$$\psi'(\xi)=\frac{1}{2}\langle Q\xi,\xi\rangle+i\langle b,\xi\rangle+\int_{z\neq 0}\Big(1-e^{-i\langle\xi,z\rangle}+i\langle\xi,z\rangle\mathbb{1}_{\{|z|\le 1\}}\Big)\nu_{X'}(\xi,dz),$$

where

$$\nu_{X'}(\xi,dz):=\nu(dz)-c|z|^{-d}f(|\xi|^2)\mathbb{1}_{\{|z|\le|\xi|^{-1}\}}dz\ge 0.$$

Therefore, Theorem 1.1 is a consequence of Proposition 2.9.

Next, we turn to consider the condition (1.3). Since

$$1-\cos z_1\ge\frac{\cos 1}{2}z_1^2\qquad\text{for } |z|\le 1,$$

we have

$$\int_{|z|\le 1}(1-\cos z_1)|z|^{-d}dz\ge\frac{\cos 1}{2}\int_{|z|\le 1}|z_1|^2|z|^{-d}dz.$$

By symmetry, for $i=1,\cdots,d$,

$$\int_{|z|\le 1}|z_1|^2|z|^{-d}dz=\int_{|z|\le 1}|z_i|^2|z|^{-d}dz=\frac{1}{d}\int_{|z|\le 1}|z|^{-d+2}dz=\frac{c_d}{d},$$

where $c_d = \pi^{d/2}/\Gamma(d/2+1)$, i.e. the volume of the unit ball in $\mathbb{R}^d$. Therefore,

$$\int_{|z|\le 1} (1-\cos z_1)|z|^{-d}dz \ge \frac{c_d \cos 1}{2d}.$$

That is, $c \le 2d/(c_d \cos 1)$. This combining with (1.3) and (3.14) gives us the required conclusion. $\square$

Having Theorem 1.1 in mind, the following condition seems to be more natural: the Lévy measure ν has *only around the origin* an absolutely continuous component, i.e. there exists $r \in (0,\infty]$ such that

$$\nu(dz) \ge |z|^{-d} f(|z|^{-2}) \mathbb{1}_{\{|z|\le r\}} dz, \tag{3.15}$$

where f is a Bernstein function. A similar lower bound condition has already been used in [20] to study gradient estimates for Ornstein-Ohlenbeck jump processes. According to [17, Corollary 4.1], cf. also the remark below Theorem 1.1, we know that under condition (3.15), the associated Lévy process X_t has the coupling property and (1.1) holds. However, the following example shows that assertion (1.4) is not satisfied.

Example 3.1. Consider the truncated rotationally invariant stable Lévy process X_t on $\mathbb{R}$ with index α. The corresponding Lévy measure is given by

$$\nu(dz) = \frac{c_\alpha}{|x|^{1+\alpha}} \mathbb{1}_{\{|z|\le 1\}},$$

where c_α is a constant depending only on α. Then, for any x, $y \in \mathbb{R}$ and $t > 0$,

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{Var}} \asymp \frac{1}{\sqrt{t}}.$$

Indeed, on the one hand, by the remark below (3.15), there exists some $C_1 > 0$ such that for any x, $y \in \mathbb{R}$ and $t > 0$, we have

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{Var}} \le \frac{C_1(1+|x-y|)}{\sqrt{t}}.$$

On the other hand, let $p_t(x,y)$ be the transition density function of X_t. According to [8, Theorem 3.6], there exist c_0, c_1, c_2, c_3 and $c_4 > 0$ such that

$$p_t(x,y) \ge \begin{cases} c_0 t^{-1/2} & t \ge R_*^\alpha,\ |x-y|^2 \le t; \\ c_1 \left(\frac{t}{|x-y|}\right)^{c_2|x-y|}, & |x-y| \ge \max\{t/C_*, R_*\}; \\ c_3 t^{-1/2} \exp\left(-\frac{c_4|x-y|^2}{t}\right), & C_*|x-y| \le t \le |x-y|^2, \end{cases} \tag{3.16}$$

where R_* and C_* are two positive constants. Denote by Z_t a truncated rotationally invariant stable Lévy process on $\mathbb{R}$ starting from 0. Then, for any x, $y \in \mathbb{R}$ with $x < y$

and $t \ge |x-y|^2 \wedge R_*^\alpha$,

$$\begin{aligned}\|P_t(x,\cdot)-P_t(y,\cdot)\|_{\mathrm{Var}} \ge & \big|\mathbb{P}\big(Z_t+x\in(x,\infty)\big)-\mathbb{P}\big(Z_t+y\in(x,\infty)\big)\big|\\ =&\big|\mathbb{P}\big(Z_t\in(0,\infty)\big)-\mathbb{P}\big(Z_t\in(x-y,\infty)\big)\big|\\ =&\mathbb{P}\big(Z_t\in(x-y,0]\big)=\int_{x-y}^0 p_t(0,z)\,dz\\ \ge&\frac{c_0(y-x)}{t^{1/2}},\end{aligned}$$

where in the last inequality we have used (3.16). The required assertion follows.

The following result is an analog of Theorem 1.1.

Theorem 3.2. *Let X_t be a Lévy process on $\mathbb{R}^d$ and ν be its Lévy measure. Assume that*

$$\nu(dz)\ge\sum_{i=1}^d\Big(|z_i|^{-1}f_i(|z_i|^{-2})\mathbb{1}_{\{z_1=\cdots=z_{i-1}=z_{i+1}=\cdots=z_d=0\}}\Big)dz, \tag{3.17}$$

where the f_i are Bernstein functions. Then, there exists $C>0$ such that for any x, $y\in\mathbb{R}^d$ and $t>0$,

$$\|P_t(x,\cdot)-P_t(y,\cdot)\|_{Var}\le 2\wedge\sum_{i=1}^d\left[\Big(\frac{|x_i-y_i|}{\sqrt{2\pi}}\int_0^\infty\frac{1}{\sqrt r}e^{-ctf_i(r)}dr\Big)\wedge\frac{C(1+|x_i-y_i|)}{\sqrt t}\right],$$

where $c=\pi^{d/2}\cos 1\big/(2d\Gamma(d/2+1))$.

Unlike (1.3) in Theorem 1.1, (3.17) is a condition on the Lévy measure restricted on the coordinate axes. Here, we mention one significant example which satisfies (3.17) (but not (1.3)).

Example 3.3. Set $L=(L^{(1)},\cdots,L^{(d)})$, where $L^{(1)}$, $L^{(2)}$, $\cdots$ $L^{(d)}$ are independent Lévy processes on $\mathbb{R}$. The Lévy measure ν of L is concentrated on the coordinate axes. Assume that ν has the following density

$$\sum_{i=1}^d\left(\mathbb{1}_{\{z_1=\cdots=z_{i-1}=z_{i+1}=\cdots=z_d=0\}}\frac{c_i}{|z|_i^{1+\alpha}}\right)dz,$$

where $c_i>0$ $(1\le i\le d)$ are constants. (Note that this measure is *more* singular than the standard rotationally invariant α-stable Lévy process.) Then, there exists a constant $C>0$ such that for any $t>0$,

$$\|P_t(x,\cdot)-P_t(y,\cdot)\|_{\mathrm{Var}}\le\frac{C|x-y|}{t^{1/\alpha}},$$

where $P_t(x,\cdot)$ is the transition function of L.

Proof of Theorem 3.2. Under condition (3.17), we can split the Lévy process X_t into

$$X_t=Y_t+Z_t,$$

where Y_t is a pure Lévy jump process with Lévy measure

$$\nu_Y(dz) := \sum_{i=1}^d \left(|z_i|^{-1} f_i(|z_i|^{-2}) \mathbb{1}_{\{z_1=\cdots=z_{i-1}=z_{i+1}=\cdots=z_d=0\}} \right) dz.$$

Z_t is independent of Y_t, and it has the Lévy measure

$$\nu_Z(dz) := \nu(dz) - \sum_{i=1}^d \left(|z_i|^{-1} f_i(|z_i|^{-2}) \mathbb{1}_{\{z_1=\cdots=z_{i-1}=z_{i+1}=\cdots=z_d=0\}} \right) dz \geq 0.$$

According to the definition of ν_Y, the generator of Y is

$$L_Y h(x) = \sum_{i=1}^d \int_{\mathbb{R}} \Big(h(x+ue_i) - h(x) - \mathbb{1}_{\{|u|\leq 1\}} u \partial_{x_i} h(x) \Big) |u|^{-1} f_i(|u|^{-2}) du,$$

where $h \in C_b^2(\mathbb{R}^d)$ and e_i is the canonical basis in $\mathbb{R}^d$. Therefore,

$$Y_t = (L^{(1)}, \cdots, L^{(d)}),$$

where $L^{(1)}$, $L^{(2)}$, $\cdots$ $L^{(d)}$ are independent one-dimensional Lévy processes with Lévy measures

$$\nu_{L^{(i)}}(du) := |u|^{-1} f_i(|u|^{-2}) du, \quad i = 1, \cdots, d,$$

respectively.

Following the proofs of Theorems 1.1 and 2.1, for $1 \leq i \leq d$, there exists a coupling $(L^{(i)}, L'^{(i)})$ of $L^{(i)}$ such that the coupling time $T^{(i)}_{x_i,y_i}$ (starting from x_i and y_i) satisfies

$$\mathbb{P}(T^{(i)}_{x_i,y_i} > t) \leq \Big(\frac{|x_i - y_i|}{2\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{r}} e^{-ctf_i(r)} dr \Big) \wedge \frac{C(1+|x_i-y_i|)}{\sqrt{t}}$$

for some constant $C > 0$ (which can be chosen independently of i). In particular, the part (Y, Y') with $Y' = (L'^{(1)}, \cdots, L'^{(d)})$ is a coupling of Y. Denote by $T_{x,y}$ the coupling time of (Y, Y'). Then, due to the independence of $L^{(1)}$, $L^{(2)}$, $\cdots$ $L^{(d)}$, we find that (see [4, Decomposition Lemma 4.18])

$$T_{x,y} = \max_{1\leq i\leq d} T^{(i)}_{x_i,y_i}.$$

Let P_t^Y be the semigroup of Y. Therefore, for any x, $y \in \mathbb{R}^d$,

$$\begin{aligned} \|P_t^Y(x,\cdot) - P_t^Y(y,\cdot)\|_{\mathrm{Var}} \leq & 2\mathbb{P}\Big(\max_{1\leq i\leq d} T^{(i)}_{x_i,y_i} > t \Big) \\ \leq & \sum_{i=1}^d \left[\Big(\frac{|x_i-y_i|}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{r}} e^{-ctf_i(r)} dr \Big) \wedge \frac{2C(1+|x_i-y_i|)}{\sqrt{t}} \right], \end{aligned}$$

where $x = (x_1, x_2, \cdots, x_d)$ and $y = (y_1, y_2, \cdots, y_d)$. Since

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{Var}} \leq \|P_t^{X'}(x,\cdot) - P_t^{X'}(y,\cdot)\|_{\mathrm{Var}},$$

as in the proof of Proposition 2.9, we are done. □

We finally turn to the proof of Proposition 1.2.

Proof of Proposition 1.2. We assume that $c = 1$ for simplicity. Let $(S_t)_{t\geq 0}$ be a subordinator associated with the Bernstein function f. For any $t \geq 0$, let μ_t^S be the transition probabilities of the subordinator S, i.e. $\mu_t^S(B) = \mathbb{P}(S_t \in B)$ for any $B \in \mathscr{B}([0,\infty))$, and $\mathbb{E}^S$ be its expectation. Then,

$$\begin{aligned}\int_0^\infty \frac{1}{\sqrt{s}} e^{-tf(s)} ds &= \int_0^\infty \frac{1}{\sqrt{s}} \int_0^\infty e^{-sr} \mu_t^S(dr) ds \\ &= \int_0^\infty \left(\int_0^\infty \frac{1}{\sqrt{s}} e^{-sr} ds \right) \mu_t^S(dr) \\ &= \int_0^\infty \frac{e^{-u}}{\sqrt{u}} du \int_0^\infty \frac{1}{\sqrt{r}} \mu_t^S(dr) \\ &= \mathbb{E}^S \left(\frac{\sqrt{2\pi}}{\sqrt{S_t}} \right).\end{aligned}$$

By using the Cauchy-Schwarz inequality twice, we arrive at

$$\int_0^\infty \frac{1}{\sqrt{s}} e^{-tf(s)} ds \geq \frac{\sqrt{2\pi}}{\mathbb{E}^S \sqrt{S_t}} \geq \frac{\sqrt{2\pi}}{\sqrt{\mathbb{E}^S S_t}}.$$

Since S_t is a Lévy process starting from 0, we easily see that $\mathbb{E}^S S_t = t\mathbb{E}^S S_1$, which yields that

$$\int_0^\infty \frac{1}{\sqrt{s}} e^{-tf(s)} ds \geq \frac{\sqrt{2\pi}}{\sqrt{t\mathbb{E}^S S_1}}. \tag{3.18}$$

We claim that $\mathbb{E}^S S_1 < \infty$ if and only if $f'(0+) < \infty$. In fact, for any $x > 0$, $\mathbb{E}^S e^{-xS_1} = e^{-f(x)}$. Then, $\mathbb{E}^S\big(S_1 e^{-xS_1}\big) = f'(x)e^{-f(x)}$. Letting $x \to 0$, by the monotone convergence theorem and the definition of Bernstein function f, we have $\mathbb{E}^S S_1 = f'(0+)$. The desired assertion follows. Therefore, if $f'(0+) < \infty$, then, due to (3.18), there exists a finite constant $C_1 > 0$ such that

$$\int_0^\infty \frac{1}{\sqrt{s}} e^{-tf(s)} ds \geq \frac{C_1}{\sqrt{t}}. \tag{3.19}$$

The proof is completed by (3.19) and Theorem 1.1. □

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